Executive summary of the Ph.D. thesis entitled

Some Analogues of Generalized Wright type Hypergeometric Function

submitted to

The Maharaja Sayajirao University of Baroda

for the partial fulfillment for the degree of

Doctor of Philosophy

in

Applied Mathematics

by

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April – 2026

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2.1 Special Functions

Special functions are the functions, which are not algebraic and cannot be represented in finitely many terms [9] (except certain cases viz. Orthogonal polynomials for e.g., Legendre, Laguerre, Hermite, Jacobi, Gegenbauer etc.) and thus they are not algebraic or elementary in nature but known as transcendental functions. In this sense exponential ($exp(x)$), logarithmic ($log(x)$), trigonometric functions ($sin(x), cos(x)$) etc. all falls under the category of Special Functions [119].

In the spirit of above understanding, many major classical special functions are being represented in terms of infinite series as solutions of differential equations [110, 119] (with variable coefficients), which either arose in the study of astrophysics and other engineering applications and are solved using power series techniques.

Generally, the functions which possess the characteristics of having their integral representations, orthogonality, recursive relationships and/or obtained as series solutions of differential equations are classified as special functions. It is important to note that, most special functions are considered as functions of complex variables; also their analyticity, singularities and cuts can be described; the differential and integral representations are known and the expansion to the Taylor series or asymptotic series are available.

2.2 History of Special Functions

Special Functions are the functions which many times arise as a series solution of certain differential equations, and which are not algebraic functions. Moreover, functions which are not algebraic but having recurrence relations and integral representations are also classified as Special Functions in the literature and are celebrated over the few decades/centuries, in respect of their occurrence in the fields of Science & Technology [12, 110].

Special functions have a long history, dating back to the early days of calculus and analysis. They became prominent in the nineteenth century, particularly within the theory of complex variables. In the later half of the twentieth century, their significance was revitalized by their connections to Lie groups and the averaging of elementary functions [9].

The study of special functions is intertwined with the advancements in terrestrial and celestial mechanics of the eighteenth and nineteenth centuries, the boundary-value problems of electromagnetism and heat in the nineteenth century, and the eigenvalue problems of quantum mechanics in the twentieth century [9].

Special functions began to emerge from seventeenth-century England. John Wallis and Newton made significant contributions, with Wallis encountering elliptic integrals and Newton developing a more sophisticated calculus.

The eighteenth century saw the discovery of the gamma function and advancements in the theory of elliptic integrals. Euler made significant contributions to the gamma function, while Legendre and Gauss made important discoveries regarding its properties. Vandermonde's theorem, Legendre polynomials, and Laplace and Legendre's addition theorem for polynomials were also notable developments.

The nineteenth century, particularly in Germany and France, marked the golden age of special functions. Developments in both mathematics and physics, such as the theory of analytic functions and field theories of physics, propelled the study of special functions forward. Elliptic functions and hypergeometric series were major areas of study during this time, with significant contributions from Jacobi, Weierstrass, Gauss, Clausen, Kummer, and others.

By the twentieth century, special functions [9] had become a vital part of both pure and applied mathematics, as well as physics. They gained added importance in physical science as solutions to the Schrödinger equation in quantum mechanics. Various generalizations and unifications of special functions were proposed, further enriching their mathematical and scientific significance.

Classical Special Functions such as the Gamma and Beta functions, Hypergeometric functions, Bessel functions , and the Riemann zeta function are not only of historical importance but also continue to serve as essential tools in contemporary scientific research. The Mittag-Leffler function has also gained prominence due to its utility in solving fractional differential equations and modeling real-world phenomena [8, 9, 12, 57, 101, 110, 131].

Among these, hypergeometric functions are particularly noteworthy due to their ability to represent many other functions through appropriate parameter choices and transformations.

2.3 Hypergeometric Functions – Historical Perspective

The historical development of hypergeometric series dates back to Wallis in the 17th century [135], with Euler contributing significantly in the 18th century. The first formal treatment was provided by Gauss in 1813 [51], leading to the classical ${}_2F_1(a, b; c; z)$ function. Later, Barnes introduced the generalized form ${}_pF_q$ in 1906 [17].

Further generalizations, such as the ${}_2R_1^{\tau}(z) \equiv {}_2R_1(a, b; c; \tau; z)$ introduced by Virchenko et al. in 2001 [133], and the more general ${}_pR_q^{\tau}(z)$ class proposed by Rai and Chauhan [28], took up the study of hypergeometric functions and their generalizations to a new height. These forms ${}_2R_1^{\tau}(z)$ and ${}_pR_q^{\tau}(z)$ allow for a broader range of functional representations and convergence behavior, thereby enhancing their applicability in both theoretical and applied settings.

In the sequel of study, Heine in 1850 [62, 63] defined and introduced q-analogue of hypergeometric functions (basic hypergeometric functions) and further extended by Jackson, Slater, Andrews, Exton, and others [43, 46, 50].

The basic hypergeometric function ${}_2\phi_1$ and the associated q-shifted factorial notation [50] connect deeply with combinatorics and orthogonal polynomial theory, with the classical hypergeometric function recovered by taking the limit $q \to 1$.

In recent decades, using the consept of zonal polynomials, the focus has also been shifted to hypergeometric functions of matrix arguments, a development initiated by Bochner [21] in 1952 and later expanded upon by Herz [64], Constantine [31], and James [72]. Further contributions by Subrahmaniam [129], Muirhead [102], Takemura [130], Gross and Richard [58], and more recently, Mathai et al. [94, 95, 97, 99], Jiu and Koutschan [74], and others [14, 15, 16, 73, 75, 76, 77, 103, 105, 106] by extending the theory to include the matrix parameters.

2.4 q-Calculus and basic (or q-) hypergeometric functions

It is known at first as, basic hypergeometric series which started essentially by Euler back in 1748 that emphasis on generating functions of partitions [43]; later, Gauss and Cauchy found several transformations and summations formulae related to basic hypergeometric series. A hundred years later after Euler discovered, Eduard Heine introduced a simple notation, known as q-number, as [62, 63]

$$[z]_q = \frac{1-q^z}{1-q}, \qquad z \in \mathbb{C}; \tag{1}$$

in which by taking limit $q \to 1$, we retrieve the complex number z.This simple equality (1), representing q-analogue of complex number z, is a gateway for q-calculus in general, where we are looking for q-analogues of mathematical objects that have the original object as limits when $q \to 1$. In 1846, Heine [62, 63] defined the basic hypergeometric series, the q-analogue of Gauss hypergeometric function, which was an innovation that became one of the most important and revolutionary developments in q-calculus.

The q-analogue of the hypergeometric function, also known as basic (or q-) hypergeometric function, is defined as below with the notations followed by Ernst [43]

$${}_2\phi_1\left(a,b;c;q,z\right) \equiv {}_2\phi_1\left[\begin{array}{c} a,b \\ \\ c \end{array};q,z\right] = \sum_{k=0}^{\infty} \frac{\langle a;q\rangle_k \langle b;q\rangle_k}{\langle 1;q\rangle_k \langle c;q\rangle_k} z^k, \tag{2}$$

where $\Re(a), \Re(b), \Re(c) > 0; z \in \mathbb{C}$. The series term in above equation converges absolutely for $|z| < 1$ when $0 < |q| < 1$. It is worth to note that the basic hypergeometric function ${}_2\phi_1(z)$ reduces to Gauss hypergeometric function ${}_2F_1(z)$ as $q \to 1$. The notation $\langle z;q\rangle_k$ in (2) is defined as below [43]:

$$\langle z;q\rangle_k \equiv (q^z;q)_k = \prod_{n=0}^{k-1}\left(1-q^{z+n}\right) \qquad z \in \mathbb{C}. \tag{3}$$

The generalized q-hypergeometric function is defined as [43]

$${}_r\phi_s(a_1,a_2,\ldots,a_r;b_1,b_2,\ldots,b_s;q,z) = \sum_{n=0}^{\infty} \frac{\langle a_1;q\rangle_n \ldots \langle a_r;q\rangle_n \cdot \left[(-1)^n q^{\binom{n}{2}}\right]^{(1+s-r)}}{\langle b_1;q\rangle_n \ldots \langle b_s;q\rangle_n \langle 1;q\rangle_n} z^n. \tag{4}$$

The series in above expression converges absolutely for all $z \in \mathbb{C}$ if $r \leq s$ and for $|z| < 1$ if $r = s+1$.

The q-binomial series is defined as in [43] as,

$$ {}_1\phi_0\,(a;-;q,z) \equiv \sum_{k=0}^{\infty} \frac{\langle a;q\rangle_k}{\langle 1;q\rangle_k} z^k = \sum_{k=0}^{\infty} \frac{(q^a;q)_k}{(q;q)_k} z^k = \frac{(q^a z;q)_\infty}{(z;q)_\infty} \quad (|z|<1)\,. \tag{5} $$

Frank Hilton Jackson (1870-1960) made significant contributions to the theory of q-calculus [29]. Most of his mathematical work was devoted to basic analogues or q-analogues (as they are sometimes called). He wrote extensively on basic hypergeometric functions [70, 69]. He introduced the q-derivative or Jackson derivative, which proved as a milestone in the study of q-calculus, which in fact is a q-analogue of the ordinary derivative [66]. Jackson also introduced the Jackson (q-) antiderivative or integration [68], q-definite integral, the q-gamma function and the q-beta function [67]. The work was subsequently greatly extended by Slater, Gasper-Rahman, Andrews, Exton, Ernst and others up to the present day [7, 42, 43, 44, 46, 50, 128].

2.5 Fractional Calculus and q-Fractional Calculus

The study of extending the concept of integer ordered derivatives and integrals to the concept of fractional order derivatives and integrals is called fractional calculus. Fractional calculus has important applications in the field of Physics and Engineering [32, 87, 88, 100, 108]. One important area of interest within this field is the investigation and derivation of fractional operators, including but not limited to the Riemann–Liouville, Weyl, Erdély-Kober left and right sided fractional operators, Caputo, Marchaud, Caputo-Fabrizio, Atangana Balenue in Caputo sense, etc.. These operators are useful to reformulate classical models in fractional models, leading to improved outcomes in various practical applications [32, 100, 108].

In 1978, Megumi Saigo [118] derived two (left and right sided) fractional integral operators involving the classical hypergeometric function with four distinct special cases. Since then, many mathematicians have contributed to the theory and applications in the area of fractional calculus involving generalized hypergeometric functions.

Related to generalized hypergeometric function ${}_2R_1^{\tau}(z)$ [133], while investigating Rao et al. [113] studied various fractional operators involving ${}_2R_1^{\tau}(z)$ and established some new results related to fractional operators involving the generalized hypergeometric function [133].

Additionally, the emergence of q-calculus has further enriched the intersection of mathematics and physics, fostering connections with quantum theory, statistical mechanics, number theory, and combinatorics [43, 46].

The concept of fractional q-calculus, an extension of ordinary fractional calculus to include q-extensions, has found widespread utility in various domains. Beginning with Al-Salam's work on the q-analogue of Cauchy's formula, researchers such as Agarwal, Isogawa, Rajkovic, Saxena, and Yadav have made significant contributions in understanding fractional q-derivatives and q-integral operators [11]. Notably, Garg and Chanchlani expanded upon Saigo's fractional operators to introduce their q-analogues [49], involving the infinite series closely related to basic hypergeometric series [50].

Note: About our contributions in this domain, details are given in Section 2 (see Chapter 4 (Section 2)).

2.6 Functions of matrix argument through zonal polynomials

Function of matrix argument, is a function in which variable $z \in \mathbb{C}$ is attempted to replace by matrix in some sense. Bessel functions of a real matrix argument were first introduced by Bochner [21] in the context of harmonic analysis and analytic number theory. Subsequently, Herz [64] provided a more detailed treatment of these functions and extended the framework by inductively defining hypergeometric functions of matrix argument through iterative applications of Laplace and inverse Laplace transforms.

Parallel developments emerged in number theory, where Olkin [104], Gindikin [55], and Kabe [79] independently studied matrix analogues of hypergeometric functions using Euler-type integral representations. For functions with complex matrix arguments, the pioneer work is due to Constantine [31] and James [72] in the domain of multivariate statistical theory and made substantial contributions by developing series expansions for hypergeometric functions of matrix arguments, often referred to as Herz-type functions.

Further exploration of these matrix functions was undertaken by Subrahmaniam [129], who revisited classical definitions and properties through zonal polynomials, and by Takemura [130], who employed integral identities and matrix transformations to investigate generalized hypergeometric functions of matrix arguments.

In 1987, Gross and Richard [58] introduced the finest structure of generalized hypergeometric functions with matrix arguments. Their work marked a significant refinement in the theory, bridging matrix-analytic techniques with advanced special function theory. Notably, when the matrix reduces to 1×1, these matrix hypergeometric functions coincides with their classical scalar counterparts.

Subsequent developments by Mathai et al. [95, 96, 97], Muirhead [102], and Jiu and Koutschan [74] enriched the theory by studying special functions of matrix arguments through zonal polynomials, which serve as symmetric homogeneous functions central to the evaluation of matrix-variate functions.

A crucial combinatorial tool in this framework is the concept of partitions. A partition κ of a non-negative integer n is given by [58, 95, 96, 102]

$$\kappa = (k_1, k_2, \ldots, k_m), \quad \text{where } k_1 \geq k_2 \geq \ldots \geq k_m \geq 0, \quad \sum_{j=1}^{m} k_j = n. \tag{6}$$

This leads to the generalized Pochhammer symbol for partitions, defined as [58, 95, 96, 97, 102]:

$$(\alpha)_\kappa = \prod_{j=1}^{m} \left(\alpha - \frac{j-1}{2} \right)_{k_j}, \quad \text{for } \Re(\alpha) > \frac{m-1}{2}. \tag{7}$$

The utility of these structures is well-established in diverse mathematical frameworks, facilitating the transition from classical scalar functions to matrix-argument functions. This transition has been especially influential in areas such as multivariate analysis, random matrix theory, and statistical inference, providing powerful tools for the study of eigenvalue distributions, matrix-variate probability models, and beyond [58, 95, 96, 97, 102, 130].

2.7 Matrix functions

Over the past two decades, the field of special functions having with parameters as matrices (known as special matrix functions), became grown remarkably, driven by both theoretical curiosity and practical need. In many complex physical problems, traditional analytical or numerical methods fall short, and this is where special matrix functions have provided elegant alternatives. Beyond physics, these functions have found their way into other areas like statistics [102, 72, 99], the theory of Lie groups [138], and both mathematical and theoretical physics

[95, 96, 97]. Their ability to capture multidimensional structures makes them especially useful in situations where scalar-based approaches are simply not enough.

At the same time, there's been a lot of attention on orthogonal matrix polynomials, which are matrix-based versions of well-known classical polynomials. Researchers have explored and developed matrix forms of Hermite, Laguerre, Chebyshev, Jacobi, Gegenbauer, Bessel, and Legendre polynomials [1, 6, 18, 23, 34, 35, 36, 37, 64, 78, 89, 120, 122, 123, 124, 132]. These matrix polynomials don't just mimic their scalar counterparts—they bring new depth, thanks to the added complexity and richness of matrix structures. They've found practical applications in solving differential equations, spectral problems, and approximation theory, offering both theoretical insights and computational tools.

One particularly fascinating direction in this broader theory is the study of matrix hypergeometric functions. When one generalizes a scalar function to multiple variables—especially in matrix form—there isn't always a single, obvious way to do it. To make such generalizations meaningful and unique, certain structural choices must be made. Still, researchers have managed to extend many classical special functions to work with vectors and matrices, while preserving important properties [46, 95, 96, 97, 99, 126]. Matrix hypergeometric functions, in particular, have become especially important in areas like multivariate statistics and random matrix theory. They serve as powerful tools for capturing complex relationships in multidimensional settings, and they continue to open up new pathways in both pure and applied mathematics.

3 Literature Survey

Ref. No.	**Author (Year)**	**Title of Paper or Book**
[1]	Abul-Dahab et al. (2015)	Reverse generalized Bessel matrix differential equation, polynomial solutions, and their properties
[3]	Al-Omari (2017)	On q-analogues of the natural transform of certain q-Bessel functions and some application

[2]	Al-Omari and Kilicman (2015)	Notes on the q-Analogues of the Natural Transforms and Some Further Applications
[4]	Albayrak et al. (2013)	On q-analogues of Sumudu transform
[5]	Albayrak et al. (2013)	On q-Sumudu Transforms of Certain q-Polynomials
[6]	Altin and Çekim (2012)	Generating matrix functions for Chebyshev matrix polynomials of the second kind
[7]	Andrews (1986)	q-Series: Their development and application in analysis, number theory, combinatorics, physics, and computer algebra
[8]	Andrews and Askey (2006)	Classical orthogonal polynomials
[9]	Andrews et al. (1999)	Special functions
[10]	Andrews (1998)	Special functions of mathematics for engineers
[11]	Annaby and Mansour (2012)	q-Fractional calculus and equations
[12]	Arfken et al. (2011)	Mathematical methods for physicists: a comprehensive guide
[13]	Atakishiyev and Atakishiyeva (2001)	A q-analogue of the Euler gamma integral
[14]	Bakhet and He (2020)	On 2-variables Konhauser matrix polynomials and their fractional integrals
[15]	Bakhet et al. (2022)	On new matrix version extension of the incomplete Wright hypergeometric functions and their fractional calculus
[16]	Bakhet et al. (2019)	On the Wright hypergeometric matrix functions and their fractional calculus
[17]	Barnes (1906)	The asymptotic expansion of integral functions defined by Taylor's series

[18]	Batahan (2006)	A new extension Hermite matrix polynomials and its applications
[19]	Bhardwaj and Sharma (2021)	An Application of q-Hypergeometric Series
[20]	Billingham et al. (2003)	Differential Equations. Linear, Nonlinear, Ordinary, Partial
[21]	Bochner (1952)	Bessel functions and modular relations of higher type and hyperbolic differential equations
[22]	Cajori (1919)	A history of mathematics
[23]	Cekim et al. (2013)	Some new results for Jacobi matrix polynomials
[24]	Charalambides (2016)	Discrete q-distributions
[25]	Chaudhary and Rao (2024)	A Note on Wright-type Generalized q-hypergeometric Function
[26]	Chaudhary and Rao (2024)	A study on unification of generalized hypergeometric function and Mittag-Leffler function with certain integral transforms of generalized basic hypergeometric function
[27]	Chaudhary and Rao (2025)	A NOTE ON HYPERGEOMETRIC FUNCTIONS OF MATRIX ARGUMENTS
[28]	Chauhan and Rai (2022)	EXTENDED GENERALIZED τ-GAUSS'HYPERGEOMETRIC FUNCTIONS AND THEIR APPLICATIONS
[29]	Chaundy (1962)	Frank hilton jackson
[30]	Ciavarella (2016)	What is q-Calculus
[31]	Constantine (1963)	Some non-central distribution problems in multivariate analysis
[32]	Das (2020)	Kindergarten of fractional calculus

[33]	Davis (1959)	Leonhard euler's integral: A historical profile of the gamma function: In memoriam: Milton abramowitz
[34]	Defez et al. (2004)	Jacobi matrix differential equation, polynomial solutions, and their properties
[35]	Defez and Jódar (1998)	Some applications of the Hermite matrix polynomials series expansions
[36]	Defez and Jódar (2002)	Chebyshev matrix polynomials and second order matrix differential equations
[37]	Defez and Tung (2017)	A new type of Hermite matrix polynomial series
[38]	Durán et al. (2014)	Rodrigues' formulas for orthogonal matrix polynomials satisfying second-order difference equations
[39]	Durán et al. (2024)	How to compute multivariate Bessel expansions
[40]	Durán and Van Assche (1995)	Orthogonal matrix polynomials and higher-order recurrence relations
[41]	Dwivedi and Sahai (2018)	On the hypergeometric matrix functions of two variables
[42]	Ernst (2003)	A method for q-calculus
[43]	Ernst (2012)	A comprehensive treatment of q-Calculus
[44]	Ernst (2018)	On Eulerian q-integrals for single and multiple q-hypergeometric series
[45]	Euler (1778)	Specimen transformations singularis serierum
[46]	Exton (1983)	Q-hypergeometric Functions and Applications
[47]	Fitouhi et al. (2006)	The Mellin transform in quantum calculus
[48]	Fitouhi et al. (2019)	On some q-versions of the Ramanujan Master Theorem
[49]	Garg et al. (2011)	q-analogue of generalized mittag-leffler function

[50]	Gasper and Rahman (2004)	Basic hypergeometric series
[51]	Gauss (1866)	Disquistiones Generales circa Seriem Infinitam 1+ α β/1. γx+ α (α+ 1) β (β+ 1)/1.2. γ (γ+ 1) xx+ α (α+ 1)(α+ 2) β (β+ 1)(β+ 2)/1.2. 3. γ (γ+ 1)(γ+ 2) xˆ 3+ etc
[52]	Geronimo (1981)	Matrix orthogonal polynomials on the unit circle
[53]	Geronimo (1982)	Scattering theory and matrix orthogonal polynomials on the real line
[54]	Geronimo and Case (1979)	Scattering theory and polynomials orthogonal on the unit circle
[55]	Gindikin (1964)	Analysis in homogeneous domains
[56]	Golub and Van Loan (1996)	Matrix computations, johns hopkins u
[57]	Gorenflo et al. (2020)	Mittag-Leffler functions, related topics and applications
[58]	Gross and Richards (1987)	Special functions of matrix argument. I. Algebraic induction, zonal polynomials, and hypergeometric functions
[59]	Guo et al. (2009)	Generalized Jacobi polynomials/functions and their applications
[60]	Gutiérrez et al. (2000)	Approximation of hypergeometric functions with matricial argument through their development in series of zonal polynomials
[61]	Hahn (1949)	Beiträge zur Theorie der Heineschen Reihen. Die 24 Integrale der hypergeometrischen q-Differenzengleichung. Das q-Analogon der Laplace-Transformation

[62]	Heine (1846)	Über die Reihe 1+(qα- 1)(qβ- 1)(q- 1)(qγ- 1) z+(qα- 1)(qα+ 1- 1)(qβ- 1)(qβ+ 1- 1)(q- 1)(q2- 1)(qγ- 1)(qγ+ 1- 1) z2+...
[63]	Heine (1847)	Untersuchungen über die Reihe
[64]	Herz (1955)	Bessel functions of matrix argument
[65]	Ismail (2005)	Classical and quantum orthogonal polynomials in one variable
[66]	Jackson (1896)	A certain linear differential equation
[67]	Jackson (1905)	A generalisation of the functions $\Gamma(n)$ and x^n
[68]	Jackson (1917)	The q-integral analogous to Borel's integral
[69]	Jackson (1942)	On basic double hypergeometric functions
[70]	Jackson (1944)	Basic double hypergeometric functions (II)
[71]	Jackson (1909)	XI.—On q-functions and a certain difference operator
[72]	James (1964)	Distributions of matrix variates and latent roots derived from normal samples
[73]	Jatav and Shukla (2022)	On Matrix Polynomials $L(M, \delta, \lambda)n(x)$
[74]	Jiu and Koutschan (2020)	Calculation and properties of zonal polynomials
[75]	Jódar and Cortés (1998)	On the hypergeometric matrix function
[76]	Jódar and Cortés (1998)	Some properties of Gamma and Beta matrix functions
[77]	Jódar and Cortés (2000)	Closed form general solution of the hypergeometric matrix differential equation
[78]	Jódar et al. (1994)	Laguerre matrix polynomials and systems of second-order differential equations
[79]	Kabe (1966)	Complex analogues of some classical noncentral multivariate distributions

[80]	Khirsariya et al. (2022)	Semi-analytic solution of time-fractional Korteweg-de Vries equation using fractional residual power series method
[81]	Khirsariya et al. (2023)	Solution of fractional modified Kawahara equation: a semi-analytic approach
[82]	Khirsariya et al. (2024)	A robust computational analysis of residual power series involving general transform to solve fractional differential equations
[83]	Khirsariya and Rao (2023)	On the semi-analytic technique to deal with nonlinear fractional differential equations
[84]	Khirsariya and Rao (2023)	Solution of fractional sawada–kotera–ito equation using caputo and atangana–baleanu derivatives
[85]	Khirsariya et al. (2023)	A novel hybrid technique to obtain the solution of generalized fractional-order differential equations
[86]	Khirsariya et al. (2024)	Investigation of fractional diabetes model involving glucose–insulin alliance scheme
[87]	Kilbas (2005)	Fractional calculus of the generalized Wright function
[88]	Kilbas et al. (2006)	Theory and applications of fractional differential equations
[89]	Kishka et al. (2012)	The generalized Bessel matrix polynomials
[90]	Kummer (1836)	Über die hypergeometrische Reihe.
[91]	Mansour (2006)	An asymptotic expansion of the q-gamma function Γ q (x)
[92]	Mansour (2008)	Some inequalities for the q-gamma function

[93]	Mansour (2009)	Linear sequential q-difference equations of fractional order
[94]	Mathai (1993)	A handbook of generalized special functions for statistical and physical sciences
[95]	Mathai (1997)	Jacobians of matrix transformation and functions of matrix arguments
[96]	Mathai and Haubold (2008)	Special functions for applied scientists
[97]	Mathai et al. (1995)	Bilinear forms and zonal polynomials
[98]	Mathai and Saxena (2006)	Generalized hypergeometric functions with applications in statistics and physical sciences
[99]	Mathai et al. (2009)	The H-function: theory and applications
[100]	Miller and Ross (1993)	An introduction to the fractional calculus and fractional differential equations
[101]	Mittag-Leffler (1903)	Sur la nouvelle fonction $E_\alpha(x)$
[102]	Muirhead (2009)	Aspects of multivariate statistical theory
[103]	Niyaz et al. (2023)	Investigation of fractional calculus for extended Wright hypergeometric matrix functions
[104]	Olkin (1959)	A class of integral identities with matrix argument
[105]	Pal et al. (2020)	Generalized Fractional Calculus Operators and the ${}_pR_q(\lambda, \eta; z)$ Function
[106]	Pal et al. (2023)	Matrix analog of the four-parameter Mittag-Leffler function
[107]	Paris (2010)	Incomplete gamma and related functions.

[108]	Podlubny (1998)	Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications
[109]	Prabhakar (1971)	A singular integral equation with a generalized Mittag-Leffler function in the kernel
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[111]	Rao (2015)	Some Studies on Wright Type Generalized Hypergeometric Function
[112]	Rao (2017)	Explicit representation and some integral transforms of sequence of functions associated with the wright type generalized hypergeometric function
[113]	Rao et al. (2013)	Some properties of generalized hypergeometric function
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[115]	Rao et al. (2013)	Wright type hypergeometric function and its properties
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[118]	Saigo (1978)	A remark on integral operators involving the Gauss hypergeometric functions
[119]	Saran et al. (2008)	Special Functions: For Mathematical, Physical & Engineering Sciences
[120]	Sayyed et al. (2004)	Gegenbauer matrix polynomials and second order matrix differential equations

[121]	Sharma and Jain (2014)	On some properties of generalized q-Mittag Leffler function
[122]	Shehata (2015)	New kinds of hypergeometric matrix functions
[123]	Shehata (2015)	Some relations on Gegenbauer matrix polynomials
[124]	Shehata (2016)	A new kind of Legendre matrix polynomials
[125]	Shukla and Prajapati (2007)	On a generalization of Mittag-Leffler function and its properties
[126]	Slater (1966)	Generalized Hypergeometric Functions
[127]	Smith (1939)	Gauss Hypergeometric Equation
[128]	Srivastava (2020)	Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis
[129]	Subrahmaniam (1976)	Recent trends in multivariate normal distribution theory: On the zonal polynomials and other functions of matrix argument
[130]	Takemura (1984)	Zonal polynomials
[131]	Titchmarsh and Heath-Brown (1986)	The theory of the Riemann zeta-function
[132]	Upadhyaya and Shehata (2011)	On Legendre matrix polynomials and its applications
[133]	Virchenko et al. (2001)	Some results on a generalized hypergeometric function
[134]	Wallis (1656)	Arithmetica Infinitorum
[135]	Wallis and Stedall (2004)	The arithmetic of infinitesimals
[136]	Watson (1910)	The continuation of functions defined by generalized hypergeometric series

[137]	Whittaker and Watson (2020)	A course of modern analysis
[138]	Willard (1968)	Lie theory and special functions
[139]	Wiman (1905)	Über den Fundamentalsatz in der Teorie der Funktionen $E_a(x)$
[140]	Wright (1933)	On the coefficients of power series having exponential singularities
[141]	Wright (1935)	The asymptotic expansion of the generalized hypergeometric function

4 Methodology

The research employs analytical and theoretical methods within classical analysis, q-calculus, and multivariate matrix theory. The methodology involves the theoretical extension and generalization of the quantum (q-) analogue of the generalized hypergeometric function, denoted as ${}_2R_1^{\tau,q}(z)$, followed by establishing its convergence criteria using rigorous analytical techniques. Furthermore, the study formulates integral representations, contiguous relations, and the n-th q-derivatives for this function. Various q-integral transforms, including q-Mellin, q-Euler (Beta), q-Laplace, q-Sumudu, and q-Natural transforms, are applied to map the function's properties and connect it to other established special functions. The research also constructs extended left- and right-sided Saigo fractional integral operators, systematically developing their q-analogues under specific parameter conditions. Finally, the methodology bridges Muirhead's matrix-variate partial differential equations and Jódar and Cortés' matrix-parameter ordinary differential equations by restricting matrix parameters to a mutually commuting family with strict positive stability bounds, thereby formulating a novel matrix partial differential equation system.

5 Key Findings

The research yields several significant theoretical advancements that unify historically separate branches of special functions. A primary finding is the successful formalization of the generalized q-hypergeometric function, proving that it seamlessly reduces to the classical generalized hypergeometric function ${}_2R_1^{\tau}(z)$ as the limit $q \to 1$. The study also establishes explicit mathematical connections between ${}_2R_1^{\tau,q}(z)$, the q-Wright function, and various q-analogues of the Mittag-Leffler functions. In the realm of fractional q-calculus, concrete summation formulae for both the left- and right-sided q-Saigo operators are derived, and the exact relations between these extended operators are established. Remarkably, the research proves that the exact solution to the newly proposed matrix partial differential equation system emerges cleanly as a novel matrix-valued hypergeometric function. This novel solution, structured entirely through Muirhead's zonal polynomials, is demonstrated to act as a master generalization. It successfully unifies the generalization of the domain, involving matrix variables, with the generalization of the coefficients, involving matrix parameters, significantly expanding the existing theoretical framework of matrix special functions.

6 Brief about our work

Recent progress in the theory of Special Functions has led to important generalizations of the hypergeometric type functions, particularly within q-calculus and their matrix analogues.

Study in the present work is about generalization ${}_2R_1^{\tau}(z)$ [133] and its q-analogue ${}_2R_1^{\tau,q}(z)$, defined in [25, 26] with

$$ {}_2R_1\left(a,b;c;\tau;z\right) = \frac{\Gamma\left(c\right)}{\Gamma\left(b\right)} \sum_{k=0}^{\infty} \frac{(a)_k \Gamma\left(b+\tau k\right)}{\Gamma\left(c+\tau k\right)} \frac{z^k}{k!}, $$

where $\Re\left(a\right), \Re\left(b\right), \Re\left(c\right); \tau \in \mathbb{R}^+, |z| < 1$ and $\Re(c-b-a) > 0$ when $|z| = 1$.

$$ {}_2{R_1}^{\tau,q}\left(z\right) \equiv {}_2R_1\left(a,b;c;\tau;q,z\right) = \frac{\Gamma_q\left(c\right)}{\Gamma_q\left(b\right)} \sum_{\ell=0}^{\infty} \frac{\langle a;q\rangle_\ell}{\langle 1;q\rangle_\ell} \frac{\Gamma_q\left(b+\tau\ell\right)}{\Gamma_q\left(c+\tau\ell\right)} z^\ell, \tag{8} $$

where $\Re\left(a\right), \Re\left(b\right), \Re\left(c\right); 0 < |q| < 1, \tau \in \mathbb{R}^+, |z| < 1$.

This function ${}_2R_1^{\tau,q}(z)$ preserves key properties while adapting to the structure of q-calculus, including convergence, differentiation, and integral representations. Further exploration on properties has revealed strong links between generalized hypergeometric ${}_2R_1^{\tau}(z)$, Wright function $W_{\lambda,\mu}(z)$ or $\phi(\lambda,\mu,z)$, and Mittag-Leffler functions of various types [125], along with their q-forms. Various q-integral transforms—such as the q-Mellin, q-Laplace, q-Euler, q-Sumudu, and q-natural—have been discussed for the newly introduced q-analogue ${}_2R_1^{\tau,q}(z)$. Employing fractional calculus in the detailed analysis for the properties, related to ${}_2R_1^{\tau}(z)$ and ${}_2R_1^{\tau,q}(z)$, has led various results on extended Saigo integral operators and their q-analogues. Moreover, primarily two analogues (matrix related) of generalized hypergeometric function ${}_2R_1^{\tau}(z)$ have been introduced, and studied, where in (i) the variable z is replaced with matrix (in essence) (see Chapter 5), followed by (ii) generalizing/replacing parameters as well as variable with the use of matrices.

7 Formation of the thesis

The work done is devised into six chapters in the present thesis. Chapter One provides the introduction and essential preliminaries for the present work. Chapter Two introduces the q-analogue of the generalized hypergeometric function ${}_2R_1^{\tau}(z)$, along with its convergence analysis and fundamental properties. Chapter Three primarily deals with the relations between the generalized hypergeometric function ${}_2R_1^{\tau}(z)$, the Mittag-Leffler functions, and the Wright function, including their respective q-analogues. Additionally, derivations involving q-integral transforms have also been incorporated in the present chapter. Chapter Four presents the extended Saigo operators, exploring their relations, and their q-versions, along with summation formulae for the associated q-integral operators. Chapter Five focuses on the generalized hypergeometric functions of matrix arguments (where in their definitions, zonal polynomials plays a pivot role), and discussed various integral properties of these functions. In Chapter Six, there introduced a novel extension of the matrix hypergeometric function, where the parameters are taken as matrices and the variable taken is considered as matrix in the sense of zonal polynomial.

Chapter 1.

Introduction and Preliminaries

"Special Function" is an age old and well-known, well explored and one of the most important branch of Mathematics, dates back to end of 18$^{\text{th}}$ century and beginning of 19$^{\text{th}}$ century. In the present chapter, historical background is given in relevance of hypergeometric functions, with some detailing about contributions of many well-known mathematicians in the field [17, 51, 135].

Apart from this historical background, as an introductory purpose, broad details related to various generalizations of hypergeometric functions, q-calculus, q-fractional calculus, functions of matrix arguments and theory of matrix special functions is made available, which are useful for the smooth development of results that have been discussed in the coming chapters.

Moreover, to acquaint the reader/researcher about fundamental and basic prerequisites, last phase of chapter–1 is made reserved for the basic definitions, results and lemmas that have been used for the later development of the theme results which are the prime aim of the present study.

Chapter 2.
Generalized q-hypergeometric function and its properties

In this chapter, the q-analogue of the generalized hypergeometric function is defined and denoted by ${}_2R_1^{\tau,q}(z)$, which reduces to the generalized hypergeometric function ${}_2R_1^{\tau}(z)$ [133], when $q \to 1$ and discussed some of its special cases. The convergence of the series, that defines the generalized q-hypergeometric function is established in two different manner, and studied various properties of it. In particular, integral representation, expression for nth q-derivative, and certain contiguous relations have been obtained. Moreover, some integral and differentiation properties also have been studied.

Chapter 3.
Relations of generalized hypergeometric functions with Mittag-Leffler and Wright functions and some results involving q-integral transforms

Along with other results this chapter highlights primarily, certain new properties of the generalized hypergeometric function ${}_2R_1^{\tau}(z)$ and its q-analogue. Interrelations among the functions ${}_2R_1^{\tau}(z)$, the Wright function, and generalized Mittag-Leffler functions have been explored. Certain results reveling connections of ${}_2R_1^{\tau,q}(z)$ with the q-Wright function and q-analogues of the different Mittag-Leffler functions have been deduced. Furthermore, various q-integral transforms of generalized q-hypergeometric function ${}_2R_1^{\tau,q}(z)$ have been obtained, which includes, the q-Mellin transform, q-Euler (Beta) transform, q-Laplace transform, q-Sumudu transform, and q-Natural transform. These results are helpful to understand the nature of hypergeometric functions in more detail.

Chapter 4.
Extended Saigo operators and their q-analogues

The Chapter 4 presents a detailed study of the extended left- and right-sided Saigo integral operators. Begin by defining these extended operators formally, followed by study of various special cases that emerge under specific parameter conditions, we discovered the relationship between the extended left- and right-sided Saigo operators. Followed by this, the present chapter includes q-analogues of the extended Saigo operators; these q-analogues are constructed in parallel to their classical counterparts and are analyzed in terms of their special cases and fundamental properties. Summation formulae for both the left and right sided q-Saigo operators are derived and studied the relation between the extended left- and right-sided Saigo q-integral operators.

Chapter 5.
Generalized hypergeometric functions of matrix argument

In this chapter, the generalized hypergeometric function of a matrix argument (through zonal polynomial) is introduced along with its multi-parameter generalization and the Fox–Wright function of matrix argument; certain special cases of them are also discussed. Various integral properties associated with the generalized hypergeometric function of matrix argument have been derived, including integral representations which can help for better understanding the structure and behaviour of these functions. In addition, Chapter also include in it some supplementary findings related to the generalized hypergeometric function ${}_2R_1^{\tau}(z)$ without matrix argument, thereby offering a broader view of its properties and theoretical implications.

Chapter 6.

Novel extension of matrix hypergeomeric function

In this chapter, the matrix Pochhammer symbol is corresponding to partition κ of $n \in \mathbb{N}$ and given matrix $F \in \mathbb{C}^{\rho\times\rho}$ is introduced and denoted by $(F)_\kappa$. Moreover multivariate gamma and beta functions of a matrix, as well as the multivariate gamma function associated with a partition κ are also introduced in terms of new definitions. In addition, hypergeometric matrix function (having parameters and variable as matrices) is defined. Moreover, the integral representation of the hypergeometric matrix function (having parameters and variable as matrices), matrix version of Binomial theorem using zonal polynomial are obtained, and the matrix form of the Gauss summation theorem is established. Several additional properties related to the hypergeometric matrix function are also deduced.

8 Overall Conclusion

In this thesis, we explored several contemporary developments in the theory of Special Functions, particularly focusing on the generalization of the classical Gauss hypergeometric function within the frameworks of q-calculus and matrix arguments. Central to our work is the study of the hypergeometric function in a generalized form and its q-analogue, denoted by ${}_2R_1^{\tau,q}(z)$, which reduces to the classical form as $q \to 1$. We investigated its structural properties, including convergence, differentiation, and integral representations.

We further examined the intricate connections between generalized hypergeometric functions, Wright functions, and Mittag-Leffler functions, including their q-analogues. The derivation of q-integral transforms such as the q-Mellin, q-Laplace, q-Euler, q-Sumudu, and q-Natural transforms provid a broader functional framework. In the study of extended Saigo integral operators and their q-analogues, by employing fractional calculus, we derived many interesting results.

Additionally, we extended the theory of hypergeometric functions to matrix arguments by utilizing zonal polynomials and matrix gamma functions. These matrix analogues, including versions of ${}_2F_1$, ${}_2R_1$, ${}_pR_q$, and ${}_p\psi_q$, were explored alongside integral representations, the matrix binomial theorem, and a matrix form of the Gauss summation theorem. These findings open up further possibilities in multivariate analysis.

9 Recommendations and Suggestions

Future research directions in the study of generalized hypergeometric functions and fractional operators present several avenues. Primary among these is the generalization of hypergeometric functions to accommodate two matrix arguments, alongside the development of efficient computational methods specifically designed for evaluating these matrix hypergeometric functions. Furthermore, detailed investigations into their theoretical properties and the exploration of q-analogues utilizing matrix arguments as both variables and parameters remain critical areas for subsequent study. Additionally, in the context of fractional calculus, there is a strong appeal to the mathematical community to further explore the intersection of traditional fractional calculus and q-calculus, leveraging these historical theoretical frameworks to address and reformulate modern application models.

Building upon these foundational matrix extensions, the culminating phase of this research established a profound unification of two historically distinct generalizations of the Gauss hypergeometric function. By simultaneously incorporating mutually commutative matrix parameters and a positive definite matrix variable, we introduced the novel multivariate matrix-valued hypergeometric function ${}_2F_1^M(\mathrm{F}_1, \mathrm{F}_2; \mathrm{F}_3; \Upsilon)$. We rigorously proved that this newly defined function is the unique analytic solution to a highly complex system of matrix partial differential equations. This differential system effectively synthesizes Muirhead's eigenvalue operators with matrix-parameter structures that were previously restricted strictly to scalar variables. Furthermore, we derived several key theoretical results for this unified function, including the matrix binomial theorem, an Euler-type integral representation utilizing the multivariate matrix beta function, and a matrix analogue of the Gauss summation theorem. These results confirm that ${}_2F_1^M$ acts as a master generalization, seamlessly collapsing into classical scalar, pure matrix-argument, or pure matrix-parameter functions under specific limiting conditions.

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